Homework 19: Intro to Algebra: p.253 #1,2, (and optionally 3)
Ling 409

(1) Consider the operation of intersection defined on some arbitrary collection of sets.
(a) Is there a two-sided identity element?
(b) Which sets have an inverse element?

(a) The universal set U is the two-sided identity element: For all sets X, $X \cap U = X$ and $U \cap X = X$.

(b) Then, given that U is the identity element, for which sets ψ is there an inverse?
i.e. for which sets ψ is there a set π for which $\psi \cap \pi = U$?
Answer: only U has an inverse: U. $U \cap U = U$. No other set has an inverse.

(2) Given an arbitrary collection of sets, what elements, if any, have inverses with respect to the operation of a) union, and b) symmetric difference? (Symmetric difference, A + B, was defined on p. 26: $A + B = (A \cup B) - (A \cap B)$, which is also equal to $(A - B) \cup (B - A)$.

In each case, as we did in problem 1, first figure out what the two-sided identity is for the given operation, if there is one. (If there isn't, then nothing will have an inverse.) (Below I use the second definition of +, only by accident; you could perfectly well use the first.)

(a) The identity for $\cup$ is $\varnothing$. The only element with an inverse with respect to $\cup$ is $\varnothing$.

(b) The (two-sided) identity element for symmetric difference is $\varnothing$:
For all A, $A+\varnothing = (A - \varnothing) \cup (\varnothing - A) = A$, and for all A, $\varnothing + A = (\varnothing - A) \cup (A - \varnothing) = A$. So the question is, for which elements A is there an inverse A^{-1}, an element that satisfies the equation $(A - A^{-1}) \cup (A^{-1} - A) = \varnothing$? There is no "recipe" for solving this, but you can reasonably suspect that the answer won't be 'random' – either only $\varnothing$ will have an inverse, or only U, or every set A. And you can suspect that the inverse for a set A, if there is one, will also not be something 'random', but will be either the set A itself or the complement of that set (U – A). So there aren't too many plausible possibilities to check.

And once you check these possibilities, you will find that every set is its own inverse under the + operation: For all A, $A + A = (A - A) \cup (A - A) = \varnothing \cup \varnothing = \varnothing$.

(3) If for a given operation in an algebra a two-sided identity exists, it is unique. Prove this for the operation of set-theoretic union.

Let's prove it in general for an arbitrary operation $\bullet$. (It would be the same for $\cup$.)
Suppose some element e is the identity element.
i.e. for all X, $e \bullet X = X$ and $X \bullet e = X$.
Suppose there is another identity element e'.
Then for all X, $e' \bullet X = X$ and $X \bullet e' = X$.
But what is the outcome of $e \bullet e'$?
Since e and e' are identity elements, the result should be *both* e and e'.
But in an algebra, the application of an operation to two arguments must be a single unique value. Therefore e and e' must be one and the same element, i.e., there can only be one identity element. (See also the proof on p. 601; slightly different, but same idea.)