Lectures 7 and 8: Infinities

Reading: Chapter 4: Infinities

4.1 Equivalent sets and cardinality

The problem: What does "the same size" mean if we want to talk about infinite sets as well as finite sets?

The solution: We generalize the notion of 'number' to the notion of 'cardinality', and we start by defining what it is for two sets to have the same cardinality.

Definition: Two sets A and B *have the same cardinality* iff there is a 1-1 correspondence between them.
Review: what does 1-1 correspondence mean?

For finite sets, "has the same cardinality as" is an *equivalence relation* that groups sets according to how many members they have. To each equivalence class we can assign a number, called the *cardinal number*, or *cardinality*, indicating the size of each set in the class. The cardinality of a finite set is always a natural number. Suppose A = {a,b,c,d}. Then |A| = 4.

For infinite sets, let's start looking at examples. I won't try to put all of this in the notes – a lot of it works better at the blackboard.

Example 1: The set P of positive integers, and the set E of even positive integers.
- Can we find a 1-1 correspondence?
- If so, same cardinality!

How shall we define *infinite*? There is more than one way, but here is one (a surprising one):

Definition: A set is *infinite* if it can be put in a 1-1 correspondence with a subset of itself.

Example 2: Show that the set N of natural numbers is infinite.

Example 3: Consider the set A* of all finite strings of letters on an alphabet A = {a,b}. We include the "empty string", which is conventionally called e.
A* = {e, a, b, aa, ab, ba, bb, aaa, ... }
-- Show that A* is infinite. Strategy: consider the subset of all strings that start with b.

Caution: We have certainly not claimed that an infinite set can be put in correspondence with <u>every</u> subset of itself. That's not true – it will never be true for any finite subset, for instance. What is true is that for any infinite set there always exists at least one subset of itself with which it can be put in 1-1 correspondence.

4.2. Denumerable sets and the cardinal Aleph-null ($\aleph_0$)

We can associate with each finite set a natural number which represents its cardinality. Sets with the same cardinality form an equivalence class.

Infinite sets can also be grouped into equivalence classes, such that all the sets in a given equivalence class have the same cardinality. (If it turned out that all infinite sets had the same cardinality, there would be just one equivalence class for all the infinite sets; but it doesn't.) We use special symbols which are not natural numbers to denote the cardinalities of infinite sets.

The most familiar infinite set is probably the set N of natural numbers {0, 1, 2, 3 ... }. The name that has been given to the cardinality of this set and all sets equivalent to it is the first letter of the Hebrew alphabet subscripted with a zero: $\aleph_0$ ("aleph-null"). $\aleph_0$ is not a natural number; it is a cardinal number which is larger than any natural number. If we ask "How many natural numbers are there?", the answer is $\aleph_0$.

Definition: A set is *denumerably infinite* (also called *denumerable,* or *countably infinite*), if it has cardinality $\aleph_0$, i.e., if it can be put in 1-1 correspondence with the set of natural numbers.
A set is called *countable* if it is either finite or denumerably infinite.

More examples of denumerably infinite sets.

(We already have: N, the set of natural numbers. E, the set of even positive integers.)

- **Z**, the set of integers, including positive, negative, and zero.
  (See 1-1 correspondence, p. 59)

- (4-4), p. 60: the set of reciprocals of the natural numbers without zero: {½, 1/3, ¼, 1/5, 1/6, ... }

- (4-5) The set of odd positive integers.

- **The set of rational numbers.** Let's do this one on the blackboard. This one is really surprising, because the rational numbers are "dense". (Remember what that means?) So shouldn't that be a "bigger" infinite set if anything is? Surprisingly, it's not.

- **General strategy:** To show that a set is denumerably infinite, find a way to put it into a 1-1 correspondence with the natural numbers. This is called *effectively listing* the members of the set, since the natural numbers themselves are thought of as forming an infinite "list".

- A "language" like that of exercise 4 on p. 71.

4.3. Nondenumerable sets.

We have seen quite a few examples of denumerably infinite sets, and we could come up with many more. (Infinitely many more, of course. "The set of all natural numbers larger than 2", "... larger than 3", "... larger than 4", etc., just for starters.) It was Georg Cantor (1845-1918), one of the main developers of set theory, who proved that the power set of any set A always has a larger cardinality than the set A itself, and therefore that there are infinitely many different infinite cardinalities. We won't try to go "up the ladder" of cardinalities (life becomes very complicated among the higher infinite cardinalities), but we will look at some sets that are larger than the set of natural numbers, and we'll learn some ways of proving that a given set is non-denumerable.

Cantor's theorem: For any set A, $|A| < |\wp(A)|$.
(Proof: see pp. 62-63. You are not responsible for knowing it. Its logic is similar to some of the things we *will* do.)

To do on the blackboard (one or both, depending on time):

Example 1: A Cantor's theorem-type example, using some infinite set and its power set. General method: a proof by contradiction.
Assume that A and $\wp(A)$ CAN be put in 1-1 correspondence. Call the correspondence F. (F is a 1-1 onto mapping from elements of A to elements of $\wp(A)$.) Then from the assumption that there is some such F, derive a contradiction.
Strategy for deriving the contradiction: consider the set $B = \{x \in A \mid x \notin F(x)\}$. Since B is a subset of A, some member y of A should be mapped onto B by F. Then ask whether $y \in B$. Now what happens?
Result: we show that there cannot be any such 1-1 onto function F. So A and $\wp(A)$ do *not* have the same cardinality. And our proof shows that it's $\wp(A)$ that's bigger.

Terminology: The cardinality of $\wp(\mathbf{N})$ must be a cardinal number greater than $\aleph_0$. The cardinality of $\wp(\mathbf{N})$ is called $2^{\aleph_0}$.
[Note: it should be an ordinary superscript, $2^{\text{aleph-null}}$, but I can't make $\aleph_0$ a superscript in Word.]

Example 2: To show that the real numbers are non-denumerable. This involves a classic *diagonal argument.*

More examples pp. 66-67:

- The set of all real numbers x, $0 \leq x < 1$, written in binary notation.
- The set of all subsets of the natural numbers, i.e. $\wp(\mathbf{N})$
- The set of all languages on a finite alphabet. (three different proofs given.)

Definition: A set which is not countable is called *uncountable* or *non-denumerable* or *non-denumerably infinite.*

4.4. Infinite vs. unbounded.

This is an issue of terminology, one on which there is often confusion. The typical case where this confusion arises is when we want to talk about the fact (or claim) that there is no longest English sentence. [Let's assume that the claim is true; we will see next time that some people have disputed it.]

"Unbounded" means "having no upper bound", and applies to some set of measurable entities. If we say that "The length of English sentences is unbounded", it means that there is no specifiable finite limit such that the length of every English sentence is under that limit.

Assume: Every sentence of English is of finite length, but there is no longest sentence.

Then:

Correct: The length of English sentences is unbounded.
Correct: English has sentences of unbounded length.
Incorrect: English has sentences of infinite length.
Correct: Every sentence of English is of finite length.
Correct: The cardinality of the set of sentences of English is infinite.
OK: The set of English sentences is infinite.
Unclear, open to both true and false interpretations: English sentences are unbounded.
(Unbounded in what respect? Need to specify "unbounded in length".)
Unclear, open to both true and false interpretations: English sentences are infinite.
(mostly false! It's not a property of individual sentences but of the *set* of all of them.)

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Suppose we have a particular dictionary that gives a large but finite list of English words.

Is the length of English sentences that do not use any word more than once bounded or unbounded?

How to avoid confusion: If possible, avoid the words *bounded* and *unbounded* altogether, and rephrase sentences involving them in terms of what it is that does or does not have a fixed upper bound.

Suggested preferred locution: There is no upper bound on the length of English sentences.

When you encounter the terms *bounded, unbounded*, check whether they are being used correctly and unambiguously. To be sure of the meaning, try to recast the given statement in a form similar to that just above.