Lectures 16 & 17: Model Theory[1]

Read: Chapter 8, Sections 5.1-5.5 (skip well-ordering); 8.5.7 (skip p. 210)

1. Theories and Models

Theories. A *theory* is a set of axioms together with all the theorems derived from them.

An alternative formulation: a *theory* is a set of statements closed under logical consequence. (A set of statements is *closed under logical consequence* iff it contains all the logical consequences of all the statements in the set.)

Models. Finding a *model* for a theory requires finding some abstract or concrete structured domain and an interpretation for all the primitive expressions of the theory in that domain such that on that interpretation all the statements in the theory come out true for that model on that interpretation.

A structured domain **M** is called a model for a theory **T** if there is a valuation [[-]] mapping the primitive expressions of **T** to **M** such that for any statement φ of **T**, $[[\varphi]]^{\mathbf{M}} = 1$.

If a theory **T** has an axiomatic characterization, **M** is a model for **T** iff it is a model for the axioms.

Some examples. Here is a not very interesting theory with just two axioms, A1 and A2. I give you also two of its theorems, T1 and T2 (without giving proofs.)

Primitive expressions: individual constant h; one-place predicate B; two-place predicate L.

(1) A1: B(h)
A2: $(\exists x)L(x,h)$

Theorems:

T1: $\neg(\forall x)\neg L(x,h)$
T2: $(\exists x)(\exists y)(B(y) \& L(x,y))$

Consider the following model **M1:** it consists of a domain D = {Harry, Susan} and the various set-theoretic constructs that can be formed from members of that set – subsets of D, sets of ordered pairs of members of D, etc.

We define the following valuation, mapping the primitive expressions of the theory to **M** as follows:

[1] This handout is based closely on one prepared by Luis Alonso-Ovalle for this course in 2003.

(2) [[h]] = Harry
[[B]] = {Harry}
[[L]] = { <Susan, Harry>, <Harry, Susan>, <Harry, Harry>}

You can verify that **M** is indeed a model for the given theory. Both axioms come out true in this model, and then so do all the theorems.

More examples: Plane geometry is a model (the 'standard model') of the Euclidean axioms. The natural numbers are the standard model of the Peano axioms. We'll see some non-standard models for the Peano axioms later – in fact you already saw one in Homework 15, where 0 is 0 but N means "is an even number" and the successor of x is x+2. All the Peano axioms are satisfied by this model as well.

We will see other examples of axioms and models in the rest of this unit, as well as in our algebra unit.

It is usually fruitful to investigate formal systems both from the perspective of axioms and from the perspective of models. Take some model that you're interested in and try to find axioms that characterize it; and tinker with sets of axioms and see what models satisfy them. You can think of linguistics as trying to characterize formally the class of possible human languages.

A word of caution. Sometimes the word "model" has a different sense, closer to what we mean by the term "theory". If you google "modelling phenomena" you can find research papers with titles like:

- *Modelling complex phenomena in speech and vision: are neural networks appropriate tools?*
- *Modelling Dynamic Phenomena in Molecular and Cellular Biology.*
- *Modelling evolutionary phenomena. Examples from genetics and game theory.*

When one speaks of *modelling* some physical phenomenon, the term usually refers to the process of building what we have called *a theory*.

PtM&W (p. 199) give a hint on how to resolve the ambiguity: look at what the model at stake is a model *of*. Models in the sense we are using are always models of axioms of other expressions in a language, they are never models of a concrete object.

2. Consistency, Completeness and Independence.

<u>Consistency</u>. A theory is consistent if it is not possible to derive from its axioms a contradiction: some statement and its denial.

Since no statement can be both true and false in a given model M, inconsistent theories have no models. That means that if you show that a theory has a model, you have shown it is consistent.

<u>Exercise</u> Do the following axioms have a model?

(3) A1: **B**(h)
A2: ¬**B**(h)
A3: (∃x)**L**(x, h)

We have introduced both a *syntactic* and a *semantic* characterization of consistency. We have said that a theory is consistent if it not possible to *derive* a contradiction from its axioms. That is a syntactic characterization. We have also said that a theory is consistent iff it has a model. That's a *semantic* characterization.

If a system is inconsistent, it's usually easier to demonstrate that syntactically than semantically: it's usually easier to derive a contradiction than to give a metalogical proof that the system has no model. But if a system is consistent, it's usually easier to demonstrate that by showing that it has a model, rather than by showing that you can't derive a contradiction from it.

Completeness.

- A formal system is called *formally complete* if every statement expressible in the system can be either proved or disproved (the negation proved) in that system (other terms for the same notion: *deductively complete, complete with respect to negation, syntactically complete*.)
- A formal system is called *semantically complete with respect to a model* M (*weakly semantically complete*) if every statement expressible in the system which is true in M is derivable in the formal system.

Independence. An axiom is *independent* if it cannot be derived from the remainder axioms in the system. A theory is said to be independent if all of its axioms are. There is also a semantic characterization of independence: a given axiom φ is independent of the other axioms of a theory **T** if **T** - {φ} has models which are not models of **T**.

Illustration. Consider again our tiny theory:

(4) A1: B(h)
A2: $(\exists x)L(x,h)$

Can you show, using the semantic characterization of independence, that A2 is independent of A1?

3. Isomorphism.

Consider the following two relations:

R1: {<a,b>, <b,c>, <c,a>}
R2: {<green, yellow>, <yellow, red>, <red, green>}

Can you say that the two relations are different? Why? Can you say that, in some sense, the two relations are the *same*? That they have the *same structure*? Why?

Isomorphism: Informally speaking, two systems are isomorphic if some specified part of their structure is identical and they differ only in interpretation or content or in unspecified parts of their structure.

The relations R1 and R2 above are isomorphic. One way to see that they are isomorphic is to see that we could "relabel" either model with a substitution of certain elements for certain other elements so that the relabeled model would then be identical to the other. The two relations are "identical up to relabeling".

Formal definition: The formal definition applies to a pair of systems A, B, each consisting of a set of elements on which one or more operations and/or relations are defined. (We will study more such systems later, in the algebra unit.)

(5) An isomorphism between such systems is a one-to-one correspondence between their elements and a one-to-one correspondence between their operations such that:

1. (a) If R holds between two elements of A, the corresponding relation R' holds between the corresponding elements of B;

   (b) if R does not hold between two elements of A, R' does not hold between the corresponding elements of B.

2. Whenever corresponding operations are performed on corresponding elements, the results are corresponding elements.

Two systems A and B are isomorphic if there exists an isomorphism between them.

For two systems not to be isomorphic, it must be the case that there is *no* isomorphism between them.

Illustration: Find an isomorphism between the two systems illustrated with R1 and R2 above. (Assume in each case that the formal system consists of a set with just the three mentioned elements and this one relation.)

4. An Elementary Formal System: Hao Wang's *L*.

Here's the set of axioms of *L*:

1. A1: $(\forall x)(x = 1 \vee x = 2 \vee x = 3)\ \&\ 1 \neq 2\ \&\ 1 \neq 3\ \&\ 2 \neq 3$
2. A2: $(\forall x)\neg R(x, x)$
3. A3: $(\forall x)(\forall y)(\forall z)((R(x, y)\ \&\ R(x, z) \rightarrow y = z)$
4. A4: $(\forall x)(\forall y)(\forall z)((R(y, x)\ \&\ R(z, x) \rightarrow y = z)$
5. A5: $(\forall x)(\exists y)R(x, y)$
6. A5: R(1, 2)

Let us first go through them and understand what they are saying.

To find a model for *L*, we need to specify a set with only three elements and an irreflexive relation which is neither one-to-many nor many-to-one such that every member of the set bears the relation to at least one other and a designated member of the set (call it 1) bears the relation to a second element (call it 2).

Here's a model:

[[1]] = a , [[2]] = b, [[3]] = c; **D** = {a, b, c}

R1: {<a,b>, <b,c>, <c,a>}

It can be shown that every model for *L* must be isomorphic to this one. If all models of a given formal system are isomorphic, the system is called *categorical.*

Since the axioms of L specify uniquely (up to isomorphism) the set D and the relation *R*, every expressible statement about D and *R* is either provable or disprovable from A1-A6, hence L is formally complete.

In the homework, and very likely in the next quiz, you will be asked to consider what happens if you change one or another of the axioms in various ways, and to test properties such as independence or non-independence of the various axioms. This formal system is simple enough to make such problems relatively manageable – it's a good testing ground for seeing if you have gotten control of the important definitions concerning formal systems and models.

5. Axioms for Ordering Relations.

Any ordering relation is a binary relation R on a set S.

(6) R is a weak partial order on S iff:

1. Transitivity: $\forall x \forall y \forall z((x \in S \ \& \ y \in S \ \& \ z \in S) \to ((Rxy \ \& \ Ryz) \to Rxz))$
2. Reflexivity: $\forall x(x \in S \to Rxx)$
3. Antisymmetry: $\forall x \forall y((x \in S \ \& \ y \in S) \to ((Rxy \ \& \ Ryx) \to x = y))$

7) R is a preorder on S iff:

1. Transitivity: $\forall x \forall y \forall z((x \in S \ \& \ y \in S \ \& \ z \in S) \to ((Rxy \ \& \ Ryz) \to Rxz))$
2. Reflexivity: $\forall x(x \in S \to Rxx)$

(8) R is a strict partial order on S iff:

1. Transitivity: $\forall x \forall y \forall z((x \in S \ \& \ y \in S \ \& \ z \in S) \to ((Rxy \ \& \ Ryz) \to Rxz))$
2. Irreflexivity: $\forall x(x \in S \to \neg Rxx)$
3. Asymmetry: $\forall x \forall y((x \in S \ \& \ y \in S) \to (Rxy \to \neg Ryx))$

(9) R is a weak linear (or total order) on S iff:

1. Transitivity: $\forall x \forall y \forall z((x \in S \ \& \ y \in S \ \& \ z \in S) \to ((Rxy \ \& \ Ryz) \to Rxz))$
2. Reflexivity: $\forall x(x \in S \to Rxx)$
3. Antisymmetry: $\forall x \forall y((x \in S \ \& \ y \in S) \to ((Rxy \ \& \ Ryx) \to x = y))$
4. Connectedness: $\forall x \forall y \ (x \neq y \to (Rxy \lor Ryx))$

(10) R is a strict linear order on S iff:

1. Transitivity: $\forall x \forall y \forall z((x \in S \ \& \ y \in S \ \& \ z \in S) \to ((Rxy \ \& \ Ryz) \to Rxz))$
2. Irreflexivity: $\forall x(x \in S \to \neg Rxx)$
3. Asymmetry: $\forall x \forall y((x \in S \ \& \ y \in S) \to (Rxy \to \neg Ryx))$
4. Connectedness: $\forall x \forall y \ (x \neq y \to (Rxy \lor Ryx))$

6. Models for Peano's axioms.

Recall Peano's axioms (N stands for "is a number" and S for "is the successor of ").

1. N0
2. $\forall x(Nx \to \exists y(Ny \ \& \ Syx \ \& \ \forall z(Szx \to z = y)))$
3. $\neg \exists x(Nx \ \& \ S0x)$
4. $\forall x \forall y \forall z \forall w((Nx \ \& \ Ny \ \& \ Sxy \ \& \ Swy \ \& \ z = w) \ ! \ x = y)$
5. If Q is a property such that
   (a) Q0
   (b) $\forall x \forall y((Nx \ \& \ Qx \ \& \ Ny \ \& \ Syx) \to Qy)$,
   then $\forall x(Nx \to Qx)$.

We can find different ("non-standard") models of these axioms.

1. Let [[0]]= 100 and let [[N]]= { 100, 101 . . . }.

2. Let [[0]]= 0 and let [[N]]= { 0, 2, 4, 6, . . . }

3. Let [[0]]= 1 and let [[N]]= {1, ½, ¼, 1/8, 1/16, … } and let [[S]]be the relation of being half of.

We can consider some interpretations of N and S that do *not* satisfy all five axioms:

1. Let [[0]]= 0 and let [[N]]= {0, 1, 2, . . . 100}. Can you see which axiom is not satisfied?

2. Let [[0]] = 0 and let [[N]]= {0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5 . . . } ([[S]] = the property of being 1 greater than.) The induction principle is violated: a property Q could satisfy the two conditions in the axiom and yet not hold for all natural numbers, by failing to hold for 0.5, 1.5, 2.5 . . .