Appendix: Euclid's Axioms

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Non-Euclidean Geometry

Introduction:

Unlike other branches of math, geometry has been connected with two purposes since the ancient Greeks. Not only is it an intellectual discipline, but also, it has been considered an accurate description of our physical space. However in order to talk about the different types of geometries, we must not confuse the term geometry with how physical space really works.

Geometry was devised for practical purposes such as constructions, and land surveying. Ancient Greeks, such as Pythagoras (around 500 BC) used geometry, but the various geometric rules that were being passed down and inherited were not well connected. So around 300 BC, Euclid was studying geometry in Alexandria and wrote a thirteen-volume book that compiled all the known and accepted rules of geometry called The Elements, and later referred to as Euclid's Elements. Because math was a science where every theorem is based on accepted assumptions, Euclid first had to establish some axioms with which to use as the basis of other theorems. He used five axioms as the 5 assumptions, which he needed to prove all other geometric ideas. The use and assumption of these five axioms is what it means for something to be categorized as Euclidean geometry, which is obviously named after Euclid, who literally wrote the book on geometry.

The first four of his axioms are fairly straightforward and easy to accept, and no mathematician has ever seriously doubted them. The first four of Euclid's axioms are:

1.) One straight line may be drawn from any two points.
2.) Any terminated straight line may be extended indefinitely.
3.) A circle may be drawn with any given center and any given radius.
4.) All right angles are congruent.

With no concern over the first four axioms, they are regarded as the axioms of all geometries or "basic geometry" for short. The fifth and last axiom listed by Euclid stands out a little bit. It is a bit less intuitive and a lot more convoluted. It looks like a condition of the geometry more than something fundamental about it. The fifth axiom is:

5.) If two straight lines lying in a plane are met by another line, and if the sum of he internal angles on one side is less than two right angles, then the straight lines will meet if the extended on the side on which the sum of the angles is less than two right angles.

The fifth axiom, also known as Euclid's "parallel postulate" deals with parallel lines, and it is equivalent to this slightly more clear statement: "For a given line and point there is only one line parallel to the first line passing through the point" (This statement was first proved to be equivalent to Euclid's fifth axiom by John Playfair in the 18th century). This seems obvious to us because of what we have been taught, but it is far less as intuitive as the first four. Later mathematicians, and even Euclid himself were not comfortable with axiom five; it is quite a complicated statement and axioms are meant to be small, simple and straightforward. Axiom five looked more like a theorem than an axiom, and as such it should have to be proved to be true and not assumed. The problem is that mathematicians were not comfortable using the fifth axiom (Even Euclid did not use it in The Elements until his 29th example). However, mathematicians found no way of showing that this problematic axiom it could be proven from the first four 4 axioms. However, all the theorems that can be proved from it worked and many mathematicians were happy just to leave it. So they came up with a version of geometry that included the fifth postulate and one that excluded it. Basic geometry was defined as being based on the first 4 axioms alone. However, Euclidean geometry was defined as using all five of the axioms.

The type of geometry we are all most familiar with today is called Euclidean geometry. Euclidean geometry consists basically of the geometric rules and theorems taught to kids in today's schools. Such as the Pythagorean theorem, rules about triangles and congruency and most other rules concerning shapes, areas, and angles. It is amazing to consider that Euclid's axioms still form the basis of our practical understanding of geometry over two thousand years later. "The Elements" had been the most widely purchased non-religious work in the world.

Introducing non-Euclidean Geometries

The historical developments of non-Euclidean geometry were attempts to deal with the fifth axiom. Mathematicians first tried to directly prove that the first 4 axioms could prove the fifth. However, mathematicians were becoming frustrated and tried some indirect methods. Girolamo Saccheri (1667-1733) tried to prove a contradiction by denying the fifth axiom. He started with quadrilateral ABCD (later called the Saccheri Quadrilateral) with right angles at A and B and where AD = BC. Since he is not using the fifth axiom, he concludes there are three possible outcomes. Angles at C and D are right angles, C and D are both obtuse, or C and D are both acute. Saccheri knew that the only possible solution was right angles. Saccheri said this was enough to claim a contradiction and he stopped. His reasoning to stop was based on faulty logic. He was going on the presumption that lines and parallel lines worked like those in flat geometry. So his contradiction was only applicable in Euclidean geometry, which was not a contradiction to what he was actually trying to prove. Of course Saccheri did not realize this at the time and he died thinking he had proved Euclid's fifth axiom from the first four. A contemporary of Saccheri, Johann Lambert (1728-1777), picked up where Saccheri left off and took the problem just a few steps further. Lambert considered the three possibilities that Saccheri had concluded as consequences of the first four axioms. Instead of finding a contradiction, he found two alternatives to Euclidean geometry. The first option represented Euclidean geometry and while the other two appeared silly, they could not be proven wrong. Through time (and quite a lot of criticism), these two other possibilities were now being considered as "alternative geometries" to Euclid's geometry. Eventually these alternate geometries were scholarly acknowledged as geometries, which could stand alone to Euclidean geometry. The two non-Euclidean geometries were known as hyperbolic and elliptic. Hyperbolic geometry was explained by taking the acute angles for C and D on the Saccheri Quadrilateral while elliptic assumed them to be obtuse. Let's compare hyperbolic, elliptic and Euclidean geometries with respect to Playfair's parallel axiom and see what role parallel lines have in these geometries:

1.) Euclidean: Given a line L and a point P not on L, there is exactly one line passing through P, parallel to L.
2.) Hyperbolic: Given a line L and a point P not on L, there are at least two lines passing through P, parallel to L.
3.) Elliptic: Given a line L and a point P not on L, there are no lines passing through P, parallel to L.

It is important to realize that these statements are like different versions of the parallel postulate and all these types of geometries are based on a root idea of basic geometry and that the only difference is the use of the altering versions of the parallel postulate. Similar to the way variations of a game are played, non-Euclidean geometries are geometries that use varying rules. Of course, when you change the rules of a game, the consequences are different and of course using varying axioms leads to different geometries. Acknowledging that there are different types of geometries is the reason we can no longer view geometry as an accurate description of our physical space. To say our space is Euclidean, is to say our space is not "curved", which seems to make a lot of sense regarding our drawings on paper, however non-Euclidean geometry is an example of curved space.

Although mathematicians showed the possibility of non-Euclidean space, people were still reluctant to reject Euclid's fifth postulate. In fact, German philosopher, Immanuel Kant, argued that since space is largely a creation of our minds, and since we cannot imagine non-Euclidean space, that Euclid's fifth postulate must necessarily be true. However, not much later, people were giving examples (models) of the non-Euclidean axiomatic systems. For instance, mathematicians long regarded a straight line as the shortest route between two points no matter what type of geometry we are considering, but when they tried this on the surface of a sphere, the arc connecting two points did not appear "straight". However appearance was not important because the lines were only curved extrinsically (as part of our perception, and not as part of a different type of geometry). Ultimately, the surface of a sphere became the prime example of elliptic geometry in 2 dimensions (although all positively curved surfaces such as a football-shaped and other elliptical objects are examples too).

Elliptic Geometry

Elliptic geometry also says that the shortest distance between two points is an arc on a great circle (the "greatest" size circle that can be made on a sphere's surface). As part of the revised parallel postulate for elliptic geometries, we learn that there are no parallel lines in elliptical geometry. This means that all straight lines on the sphere's surface intersect (specifically, they all interesect in two places). A famous non-Euclidean geometer, Bernhard Riemann, who dealt mostly with and is credited with the development of elliptical geometries, theorized that the space (we are talking about outer space now) could be boundless without necessarily implying that space extends forever in all directions. This theory suggests that if we were to travel one direction in space for a really long time, we would eventually come back to where we started! This theory involves the existence of four-dimensional space similar to how the surface of a sphere (which is three dimensional) represents an elliptic 2 dimensional geometry. Einstein addressed the idea that space could be unbounded without being in finite in his theory of relativity. However, it should be noted that this idea raises some issues regarding Euclid's second axiom, which says a line segment can be extended indefinitely.

There are many practical uses for elliptical geometries. Elliptical geometry, which describes the surface of a sphere, is used by pilots and ship captains as they navigate around the spherical Earth, which we live. In fact, working in elliptical geometry has some non-intuitive results. For example, the shortest flying distance from Florida to the Philippine Islands is a path across Alaska. The Philippines are South of Florida so it is not apparent why flying North to Alaska would be shorter. The answer is that Florida, Alaska, and the Philippines are collinear locations in elliptical geometry. Another odd property of elliptical geometry is that the sum of the angles of a triangle is always greater then 180°. Relatively small triangles, such as a triangle that is formed by three intersecting roads have angle sums very close to 180°. In order to notice the effect of triangles with larger angle sums, we have to consider much larger triangles such as the triangle formed by New York, Los Angeles and Miami. Because a triangle's angle sum distortion is proportional to the size of the triangle, we also can deduce that a triangle's area is related to its angle sum! Furthermore, this notion destroys our idea of similar triangles, because similar triangles are triangles with the same angle measurments but different areas but since area is related to angle sum in elliptic geometry, there are no similar triangles (just congruent ones)!

Hyperbolic Geometry:

The other type of non-Euclidean geometry we have yet to examine is hyperbolic geometry. Recalling the corresponding Playfair's axiom for hyperbolic geometry, we see that in hyperbolic geometry, there is more than one parallel line to L, passing through point P, not on L. Furthermore, hyperbolic geometries comes with some more restrictions about parallel lines. In Euclidean geometry, we can show that parallel lines are always equidistant, but in hyperbolic geometries, of course, this is not the case. Therefore, in hyperbolic geometries, we merely can assume that parallel lines carry only the restriction that they don't interesect. Furthermore, the parallel lines don't seem straight in the conventional sense. They can even approach each other in an asymptotically fashion. The surfaces on which these rules on lines and parallels hold true are on negatively curved surfaces.

When compared with the other geometry's triangle angle sums, we see that in hyperbolic geometry, the triangle's angle sum is less than 180 degrees whereas elliptic geomtry has more than 180 degrees. Similarly, the larger the sides of the triangle, the greater the distortion of the angle sums on both elliptic and hyperbolic geometries. Much like elliptic geometries, the area of a triangle is proportional to its angle sum and of course this implies that there are no similar triangles as well. In some ways, hyperbolic geometry is simpler than elliptic so technically hyperbolic was discovered first. Gauss, Schweikart, Lobachevsky and János Bolyai all separately and all in the first half of the 19th century, are credited with the discovery of hyperbolic geometry. Now that we see what the nature of a hyperbolic geometry, we probably might wonder what some models of hyperbolic surfaces are. Some traditional hyperbolic surfaces are that of the saddle (hyperbolic parabola) where the surface curves in two different directions and more scholarly, the Poincaré Disc. The Poincaré Disc is a model of hyperbolic geometry envisioned by French mathematician/philosopher, Poincaré (1854 – 1912). His model is a sort of 2 dimensional model, which makes it appealing to those who are working on paper. In one of his philosophical writings, Science and Hypothesis 1901, he wrote of his model as an imaginary universe occupying the interior of a disc (or circle) in the Euclidean plane. And we as observers get to watch the inhabitants move around. However, they appear to "shrink" as they approach the infinitely distant horizon (the boundary of the disc). Furthermore, the inhabitants do not notice the effect because their ruler shrinks with them as they move. They think that they live in a normal Euclidean space, but we see them in a non-Euclidean space with their dimensions behaving strangely. Since the edge of the disc represents infinity, their universe still contains infinite space, however large line segments appear to grow smaller as they get closer to the circle's edge. Straight lines, in the Poincaré Disc, intersect the disc's edge at 90-degree angles. Like many of our examples of non-Euclidean geometries, measurements on the Poincaré Disc become more distorted when we are looking at larger areas and line segments. In fact, if you were to draw a triangle with vertices close to the edge of the disc (infinity), the triangle's area would be near zero!

Applications of non-Euclidean Geometries:

Practically, non-Euclidean geometries have long been regarded as "curiousities" because they seemed to have little to do with our real universe. Thanks to Einstein and subsequent cosmologists, non-Euclidean geometries began to replace the use of Euclidean geometries in many contexts. For example, physics is largely founded upon the constructs of Euclidean geometry but was turned upside-down with Einstein's non-Euclidean "Theory of Relativity" (1915). Newtonian physics, based upon Euclidean geometry, failed to consider the curvature of space, and that this constituted for major errors in the equations of planetary motion and gravity.

Einstein's general theory of relativity proposes that gravity is a result of an intrinsic curvature of spacetime (as opposed to a Newtonian action-at-a-distance explanation). Intrinsic curvature, explains how straight lines could have the properties associated with curvature without actually being curved in the ordinary sense, is now used to explain how something which is obviously curved, like the orbit of a planet, is really straight. Also, it means that if we accept an intrinsic curvature of spacetime (a curve of spacetime, as opposed to a curve within spacetime), then these "curved lines" in three dimensional space (such as those used to describe gravity), then we must assume that the lines are curved in a higher dimension, into which the straight lines are curved in the conventional sense. In layman's terms, this explains that the phrase "curved space" is not a curvature in the usual sense but a curve that exists of spacetime itself and that this "curve" is in the direction of the fourth dimension.

So if our space has a non-conventional curvature in the direction of the fourth dimension, that that means our universe is not "flat" in the Euclidean sense and finally we realize our universe is probably best described by a non-Euclidean geometry! The future of the universe will be determined by whatever is the geometry of the universe happens to be. According to current theories in cosmology, if the geometry is hyperbolic, the universe will expand indefinitely; if the geometry is Euclidean, the universe will expand indefinitely at escape velocity; and if the geometry is elliptic, the expansion of the universe will coast to a halt, and then the universe will start to shrink back to a singularity and possibly to explode again with a whole new big bang.

Conclusion:

It would be fallacious to assume that because math "works" that we understand what is happening in our real space. However there are those that speak as though math somehow explains what is going on in the real world. The mathematics of Euclid were were simple and straightforward, but it did not confer an understanding about what the nature of the universe was. Many people who, with an almost religious fervor, proclaimed that Euclidean geometry was the one and only geometry resisted the recognition of the existence of the non-Euclidean geometries as mathematical systems. Such attitudes reflect a failure to recognize that geometry is a mathematical system that is determined by its assumptions. The most we can assume about our universe now is that Euclidean geometry provides an excellent representation for the localized part of the universe that we inhabit. Poincaré added some insight to the debate between Euclidean and non-Euclidean geometries when he said, "One geometry cannot be more true than another; it can only be more convenient".